

We give solutions to some problems from our earlier conjecture collection[S].

In the following, the boxes contain the number of the conjecture with the OEIS sequence number[OEIS].

1 / A002421

$$(1 - 4x)^{3/2} = 1 - 6x + \sum_{n>1} \frac{12(2n-4)!}{n!(n-2)!} x^n.$$

Proof.

$$\begin{aligned} [x^n](1 - 4x)\sqrt{1 - 4x} &= \frac{\binom{2n}{n}}{1 - 2n} - 4 \frac{\binom{2n-2}{n-1}}{3 - 2n} \\ &= \frac{4n^2(2n-1)(2n-2)! - (2n-3)(2n)!}{(2n-1)(2n-3)n!^2} \\ &= \frac{4n^2(2n-1)(2n-2)(2n-4)! - (2n)!}{(2n-1)n!^2} \\ &= \frac{4n^2(2n-2)(2n-4)! - 2n(2n-2)!}{n!^2} \\ &= \frac{(2n-4)! [4n^2(2n-2) - 2n(2n-2)(2n-3)]}{n!n(n-1)(n-2)!} \end{aligned}$$

□

2 / A071721

With $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$,

$$(1 + x^2 C(x)^2) C(x)^2 = \sum_n \frac{6n(2n)!}{n!(n+1)!(n+2)!} x^n.$$

Proof.

$$\begin{aligned} C(x)^2 + x^2 C(x)^4 &= \frac{2 - 4x - 2\sqrt{1 - 4x}}{4x^2} + x^2 \frac{8 - 16x - 4(2 - 4x)\sqrt{1 - 4x}}{16x^4} \\ &= \frac{1 - 2x + (x-1)\sqrt{1 - 4x}}{x^2}. \end{aligned}$$

$$\begin{aligned} [x^n](-1 + x)\sqrt{1 - 4x} &= -\frac{\binom{2n}{n}}{1 - 2n} + \frac{\binom{2n-2}{n-1}}{3 - 2n} \\ &= \frac{(2n-3)(2n)! - n^2(2n-1)(2n-2)!}{(2n-1)(2n-3)n!^2} \end{aligned}$$

and so on, finally substitute $n + 2$ for n .

□

104 / A020330

$a_n =$ binary of n twice juxtaposed \iff
 $\{a_1 = 3, a_{2n} = 4a_n - 2n, a_{2n+1} = 4a_n - 2n + 2^{\lfloor \lg 4^{n+2} \rfloor} + 1\}$

Proof. The binary of n twice juxtaposed is simply n plus ' n shifted left by k bits', where k is the binary length of n , so we have

$$a_n = n + n2^{\lfloor \lg n \rfloor + 1},$$

leading to

$$a_{2n} = 2n + 2n2^{\lfloor \lg 2n \rfloor + 1} = 2n + 4n2^{\lfloor \lg n \rfloor + 1} = 4a_n - 2n,$$

$$a_{2n+1} = 2n + 1 + (2n + 1)2^{\lfloor \lg 2^{n+1} \rfloor + 1} = \text{etc.}$$

□

108 / A006582

$$a_n = \sum_{k=1}^{n-1} k \text{ XOR } n - k \iff \{a_{<2} = 0, a_{2n} = 2a_{n-1} + 2a_n + 4n - 4, a_{2n+1} = 4a_n + 6n\}$$

Proof.

$$\begin{aligned} a_{2n} &= \sum_{k=1}^{2n-1} k \text{ XOR } (2n - k) = \sum_{k=1}^{n-1} 2k \text{ XOR } (2n - 2k) + \\ &\quad + \sum_{k=1}^{n-1} 2k + 1 \text{ XOR } (2n - 2k - 1) + (1 \text{ XOR } 2n - 1) \\ &= 2 \sum_{k=1}^{n-1} k \text{ XOR } n - k + \left[\sum_{k=1}^{n-1} 2k + 1 \text{ XOR } 2(n-1-k) + 1 \right] + 2n - 2 \\ &= 2a_n + 2 \sum_{k=1}^{n-1} k \text{ XOR } (n-1-k) + 2n - 2 \\ &= 2a_n + 2a_{n-1} + 2(n-1 \text{ XOR } 0) + 2n - 2. \end{aligned}$$

$$\begin{aligned} a_{2n+1} &= \sum_{k=1}^{2n} k \text{ XOR } (2n + 1 - k) = \\ &= \sum_{k=1}^{n-1} 2k \text{ XOR } (2n + 1 - 2k) + \sum_{k=1}^{n-1} 2k + 1 \text{ XOR } (2n + 1 - 2k - 1) \\ &\quad + (1 \text{ XOR } 2n) + (2n \text{ XOR } 1) \\ &= \left[n - 1 + 2 \sum_{k=1}^{n-1} k \text{ XOR } n - k \right] + \left[n - 1 + 2 \sum_{k=1}^{n-1} k \text{ XOR } n - k \right] \\ &\quad + 2(2n + 1) = 4a_n + 6n. \end{aligned}$$

□

109 / A006583

$$a_n = \sum_{k=1}^{n-1} k \text{ OR } n - k \iff \{a_{<2} = 0, a_{2n} = 2a_{n-1} + 2a_n + 5n - 4, a_{2n+1} = 4a_n + 6n\}$$

Proof.

$$\begin{aligned}
a_{2n} &= \sum_{k=1}^{2n-1} k \text{ OR } (2n - k) = \sum_{k=1}^{n-1} 2k \text{ OR } (2n - 2k) + \\
&\quad + \sum_{k=1}^{n-1} 2k + 1 \text{ OR } (2n - 2k - 1) + (1 \text{ OR } 2n - 1) \\
&= 2 \left[\sum_{k=1}^{n-1} k \text{ OR } n - k \right] + \left[\sum_{k=1}^{n-1} 2k + 1 \text{ OR } 2(n-1-k) + 1 \right] + 2n - 1 \\
&= 2a_n + n - 1 + 2 \sum_{k=1}^{n-1} k \text{ OR } (n - 1 - k) + 2n - 1 \\
&= 2a_n + n - 1 + 2a_{n-1} + 2(n - 1 \text{ OR } 0) + 2n - 1.
\end{aligned}$$

$$\begin{aligned}
a_{2n+1} &= \sum_{k=1}^{2n} k \text{ OR } (2n + 1 - k) = \\
&= \sum_{k=1}^{n-1} 2k \text{ OR } (2n + 1 - 2k) + \sum_{k=1}^{n-1} 2k + 1 \text{ OR } (2n + 1 - 2k - 1) \\
&\quad + (1 \text{ OR } 2n) + (2n \text{ OR } 1) \\
&= \left[n - 1 + 2 \sum_{k=1}^{n-1} k \text{ OR } n - k \right] + \left[n - 1 + 2 \sum_{k=1}^{n-1} k \text{ OR } n - k \right] \\
&\quad + 2(2n + 1) = 4a_n + 6n.
\end{aligned}$$

□

References

- [OEIS] N. Sloane, *Online Encyclopedia of Integer Sequences*, www.research.att.com/~njas/sequences/Seis.html
- [S] R. Stephan, *Prove or disprove. 100 Conjectures from the OEIS*, preprint, [math.CO/0409509](https://arxiv.org/abs/math/0409509)